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Absence of Absolutely Continuous Spectrum of Floquet Operators

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The spectrum of the Floquet operator associated with time-periodic perturbations of discrete Hamiltonians is considered. If the gap between successive eigenvalues λ_j of the unperturbed Hamiltonian grows as $\lambda_j - \lambda_{j-1} \simeq j^\alpha$ and the multiplicity of λ_j grows as j^β with $\alpha > \beta \geqslant 0$ as j tends to infinity, then the corresponding Floquet operator possesses no absolutely continuous spectrum provided the perturbation is smooth enough.

KEY WORDS: Time-periodic Hamiltonians; spectrum of Floquet operators; quantum stability; perturbation of dense point spectra.

1. INTRODUCTION

Consider a positive self-adjoint Hamiltonian H_0 on a separable Hilbert space $\mathscr{H}$ with discrete spectrum $\{\lambda_j\}_{j=1,\dots,\infty}$ and $W(t)$ a symmetric time-dependent periodic perturbation

$$W(t+2\pi) = W(t), \qquad \forall t \in \mathbf{R} \tag{1.1}$$

The associated Floquet operator, defined by

$$F = -i\frac{\partial}{\partial t} + H_0 + W(t) \tag{1.2}$$

on $L^2[0, 2\pi] \otimes \mathscr{H}$ with periodic boundary conditions in t, has been the object of considerable interest recently. More precisely, the nature of the spectrum of F, $\sigma(F)$, has been investigated thoroughly for specific models

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and for more general situations as well, as reviewed by Bellissard[1] and more recently by Jauslin.[12] The interest of such detailed studies for physics lies in the fact that the long-time behavior of the solutions $\psi(t)$ of the time-dependent Schrödinger equation

$$i\frac{\partial}{\partial t}\psi(t)=(H_0+W(t))\,\psi(t), \qquad \psi(0)=\varphi \tag{1.3}$$

is closely related to the spectral properties of the Floquet operator. This asymptotic behavior of the solutions of (1.3) is quite relevant for the study of quantum stability or quantum chaos.[1,12] Let $U(t)$ be the unitary evolution associated with (1.3) such that

$$\psi(t)=U(t)\varphi \tag{1.4}$$

As a consequence of the periodicity of the Hamiltonian $H_0+W(t)$, the solution of (1.3) satisfies

$$\psi(n2\pi)=U(2\pi)^n\,\varphi, \qquad \forall n\in\mathbf{N} \tag{1.5}$$

where $U(2\pi)$ is the monodromy operator. The large-n behavior of $\psi(n2\pi)$ is thus clearly dependent on the spectral subspace (pure point, absolutely or singular continuous) of $U(2\pi)$ in which the initial condition φ is chosen, as discussed in refs. 1 and 12. On the other hand, the spectral properties of the Floquet operator F and of the monodromy operator $U(2\pi)$ are equivalent, as established in refs. 20 and 7. Note that this type of approach can be generalized in order to study the stability of the quantum dynamics when the perturbation $W(t)$ is quasiperiodic in time, as demonstrated recently in ref. 3.

From the mathematical point of view, these considerations pertain to the perturbation theory of operators with dense pure point spectrum. Indeed, if $W(t)\equiv 0$, the spectrum of the Floquet operator is given by $\{n+\lambda_j\}_{j=1,\dots,\infty}^{n\in\mathbf{Z}}$, which is generically dense on the real line. Correspondingly, the spectrum of the associated monodromy operator $U_0(2\pi)$ consists of the set $\{e^{-2\pi i\lambda_j}\}_{j=1,\dots,\infty}$, which generically fills the unit circle densely. The question is the following: does this pure point spectrum remain stable after perturbation by the time-periodic operator $W(t)$? Although this problem is in general rather delicate,[8] a rigorous complete positive answer can be given for certain specific models, in some range of parameters, for example, the pulsed rotor considered by Bellissard,[1] some time-dependent quadratic Hamiltonians studied by Hagedorn *et al.*,[6] a class of time-dependent perturbations of the harmonic oscillator, and discrete Hamiltonians kicked periodically by some rank-one perturbations as shown by Combescure.[4,5] On the other hand, results valid for general systems are scarce and they provide only a partial answer to the above question. These results are based on the search for general criteria allowing one to exclude the presence of absolutely continuous spectrum in the Floquet operators. This approach was initiated and developed by Howland in a series of papers.[9-11] Such results yield a partial answer to our question in the sense that they say nothing about a singular continuous spectrum. However, the absence of an absolutely continuous component $\sigma_{ac}(F)$ in the spectrum of F is sometimes already considered as a stability result. The criteria obtained are of the following form. Assume the Hamiltonian H_0 has simple eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_j < \cdots$ satisfying the growth condition

$$\lambda_j - \lambda_{j-1} \simeq j^\alpha \tag{1.6}$$

for some $\alpha > 0$ and suppose that $W(t)$ is uniformly bounded. Then it is proven in ref. 10 that $\sigma_{ac}(F) = \varnothing$ for any $\alpha > 0$ provided $W(t)$ is smooth enough. Actually, this result is also true for some class of unbounded perturbations $W(t)$, as discussed in ref. 11, and it is even shown in ref. 9 that $\sigma(F)$ is generically dense pure point, in an appropriate probabilistic sense, provided $\alpha > 2$. However, the restriction imposed on the eigenvalues of H_0 to be nondegenerate was conjectured in ref. 10 to be technical only. Indeed, the result is expected to hold for the pulsed rotor of Bellissard, a case where the corresponding eigenvalues satisfy (1.6) but are doubly degenerate. Accordingly, it was proven recently by Nenciu[17] that $\sigma_{ac}(F) = \varnothing$ if the growth condition (1.6) is satisfied, and if the multiplicity m_j of the eigenvalues λ_j is uniformly bounded in j, provided $W(t)$ is smooth enough. However, to prove this result, it was necessary to impose another technical condition, namely $\alpha > 1/2$.

In this paper we reconsider the absence of absolutely continuous spectrum of Floquet operators in the generalization due to Nenciu[17] of the framework designed by Howland[9-11] (see below). We improve the preceding results in two ways. First, we remove the technical condition $\alpha > 1/2$ to give a complete proof of the conjecture of Howland[10] in the degenerate case for any positive α. Second, and more important, we can allow the degeneracy m_j of the eigenvalues λ_j to increase with j as $m_j \simeq j^\beta$, with $\beta < \alpha$. This generalization may be of interest for the study of systems of more than one degree of freedom since in such cases the degeneracy is likely to increase with the principal quantum number. See, however, the remarks following the main Theorem 2.1.

2. RESULT AND STRATEGY

Let H_0 be an operator satisfying the following spectral hypothesis S:

1. H_0 is densely defined on $D \subset \mathscr{H}$, self-adjoint, and positive.
2. $\sigma(H_0) = \bigcup_{j=1}^{\infty} \sigma_j$.
3. σ_j consists of a finite number of finitely degenerate eigenvalues such that:

   (a) $\max_{\lambda, \mu \in \sigma_j} |\lambda - \mu| \leqslant c_0$.

   (b) $\operatorname{dist}(\sigma_j, \sigma_{j-1}) \geqslant c_1 j^{\alpha}$, $\alpha > 0$.

   (c) $\operatorname{mult}(\sigma_j) \leqslant c_2 j^{\beta}$, $\beta \geqslant 0$.

Here c_0, c_1, c_2, α, and β are independent of j.

Let $W(t)$ be an operator satisfying the regularity condition R_k:

1. $W(t)$ is bounded and symmetric $\forall t \in \mathbf{R}$.
2. $W(t)$ is strongly C^k, $\forall t \in \mathbf{R}$.

The operator $H_0 + W(t)$ is thus self-adjoint, densely defined on D (ref. 15, Theorem 4.3, p. 287), and strongly C^k, with bounded derivatives. If $k \geqslant 1$, there exists a unitary evolution operator $U(t)$, strongly C^1 on D, which maps D into D and satisfies for any φ in D and $t \in \mathbf{R}$

$$i \frac{\partial}{\partial t} U(t)\varphi = (H_0 + W(t))\, U(t)\varphi, \qquad U(0) = \mathbf{I} \tag{2.1}$$

as can be deduced from Theorem X70.[19]

Theorem 2.1. Let H_0 satisfy S and $W(t)$ be 2π-periodic in t and satisfy R_k, with $k \geqslant 1$. If $\alpha > \beta \geqslant 0$ and $k \geqslant [(1+\beta)/\alpha] + 1$, then $\sigma_{\mathrm{ac}}(U(2\pi)) = \varnothing$.

Remarks. Setting $\beta = 0$, we obtain the conjecture of Howland[10] for any $\alpha > 0$.

It is possible to weaken the spectral hypothesis S somehow since the result also holds if the size of the spectral sets σ_j grows as

$$\max_{\lambda, \mu \in \sigma_j} |\lambda - \mu| \leqslant c_0 j^{\alpha} \tag{2.2}$$

As already noticed, the above theorem describes $\sigma(U(2\pi))$ only partially. However, the result is not of perturbative nature, in the sense that the norm of the operator $W(t)$ can be arbitrarily large. By contrast, the complete characterization of $\sigma(F)$ performed by Bellissard[1] and Combescure[4] on their models can be achieved in regimes where $\| W(t) \|$ is small enough.

This result can also be useful for some cases where $W(t)$ is unbounded. Indeed, it is shown in ref. 11 how to reduce the study of the Floquet operator $-i\,\partial/\partial t + H_0 + W_u(t)$, where $W_u(t)$ belongs to a certain class of unbounded operators, to the study of $-i\,\partial/\partial t + H_0 + W_b(t)$, where $W_b(t)$ is bounded. And, according to the final remark of §2 in ref. 11, a similar procedure can be applied when H_0 has degenerate eigenvalues.

As noted in ref. 12, this type of result is likely to apply to one-degree-of-freedom systems essentially, because of the growth condition on the gaps between successive eigenvalues in hypothesis S. This impression is strengthened by the supplementary condition imposed on the growth of the multiplicity of eigenvalues. Indeed, consider the simple two-degree-of-freedom system given by a free rotator in $\mathbf{R}^3$. The Hamiltonian of the system is $H_0 = \mathbf{J}^2$, where $\mathbf{J}$ denotes the angular momentum operator. The gaps between the eigenvalues $\lambda_j = j(j+1)$ of H_0 do satisfy the growth condition with exponent $\alpha = 1$,

$$j(j+1) - (j-1)j = 2j \tag{2.3}$$

However, the multiplicity of these eigenvalues grows as

$$\mathrm{mult}(\lambda_j) = 2j + 1 \tag{2.4}$$

so that the exponent $\beta = 1$ and the condition $\alpha > \beta$ is not satisfied. This is nevertheless in agreement with the fact that multidimensional systems seem to be more inclined to instabilities.[12] Note that if we consider formally $H_0 = \mathbf{J}^4$, the corresponding eigenvalues $\lambda_j = j^2(j+1)^2$ and their multiplicities behave in the proper way with exponents $\alpha = 3$ and $\beta = 1$.

The strategy followed to prove results of this type, which is common to refs. 10 and 17 and the present work, is based on the few general theorems on the stability of the absolutely continuous spectrum. In ref. 10, where the Floquet operator F is considered, Howland shows by means of a KAM-inspired procedure that F is unitarily equivalent to an operator $F_0 + R$, where F_0 is self-adjoint and has a pure point spectrum and R is trace class. Then it remains to invoke the Kato–Rosenblum theorem (ref. 15, Theorem 4.4, p. 540) stating that the absolutely continuous subspaces of self-adjoint operators are unitarily equivalent if they differ by a trace-class operator. Nenciu[17] deals with the monodromy operator $U(2\pi)$ instead of F, and uses recently developed tools in the adiabatic theory[18] to approximate $U(2\pi)$ by $V + R$, where V is a unitary operator having pure point spectrum and R is trace class. The result is thus achieved by virtue of the Birman–Krein theorem,[2] an equivalent of the Rosenblum–Kato theorem for unitary operators, which assesses that the absolutely continuous spectral subspaces of unitary operators are unitarily equivalent if they differ by a trace-class operator. Our proof of Theorem 2.1 follows the latter method proposed by Nenciu; however, we make use of another adiabatic approximation technique developed in refs. 13 and 14. In the present context, where no small parameter appears, this method proves to be very efficient as well. Indeed, its relative simplicity makes it possible to obtain the accurate estimates which are needed to bound operators in the trace norm and to extend the previous results as described in Theorem 2.1.

3. PROOF

We present in this section the proof of Theorem 2.1 based on the adiabatic techniques developed in ref. 14. In doing so, we make use of some intermediate results which will be proven in the next technical section.

3.1. Preliminaries

Let us first consider the stability of the spectral hypothesis S.

Lemma 3.1. Let H_0 satisfy S and let $B(t)$ satisfy R_0. Then $H_0+B(t)$ satisfies S uniformly in $t\in[0, 2\pi]$ with the same exponents α and β.

Proof. The uniform boundedness of $B(t)$ and the growth condition on the gaps ensure that $\sigma(H_0+B(t))$ consists of the disjoint union of new sets $\bigcup_{k=1}^{\infty}\sigma'_k(t)$. Hence, taking into account a possible relabeling, the gaps between these sets behave as

$$\mathrm{dist}(\sigma'_k(t), \sigma'_{k-1}(t))\geqslant c_1(k+r)^\alpha\geqslant c'_1 k^\alpha \tag{3.1}$$

for large k. Moreover, it is readily seen by considering the interpolating operator $H_0+xB(t)$, where x ranges in $[0, 1]$, that

$$\mathrm{mult}(\sigma'_k(t))=\mathrm{mult}(\sigma_{k+r})\leqslant c'_2 k^\beta \tag{3.2}$$

for k large enough. ∎

Notation. There will appear several constants in the sequel which we shall denote generically by the same symbol c. From now on, the time derivative will be denoted by a prime.

Consider the operator $H_0+B(t)$, where H_0 satisfies S and $B(t)$ satisfies R_0. Let $R(t, \lambda)=(H_0+B(t)-\lambda)^{-1}$. By the above lemma, for $t\in[0, 2\pi]$, the spectrum $\sigma(t)$ of $H_0+B(t)$ consists of spectral sets $\sigma_j(t)$ and the associated spectral projectors $P_j(t)$ can be written as

$$P_j(t)=-\frac{1}{2\pi i}\oint_{\Gamma_j} R(t, \lambda)\, d\lambda \tag{3.3}$$

where the nonintersecting paths Γ_j surrounding $\sigma_j(t)$ are chosen in such a way that

$$\operatorname{long}(\Gamma_j) \equiv |\Gamma_j| \leqslant cj^{\alpha} \tag{3.4}$$

and

$$\operatorname{dist}(\Gamma_j, \sigma(t)) \geqslant cj^{\alpha} \tag{3.5}$$

Remark. Both estimates (3.4) and (3.5) are true for appropriate paths Γ_j if the length of the sets $\sigma_j(t)$ grows as j^{α}, in the spirit of the first remark below Theorem 2.1.

Proposition 3.1. Let H_0 satisfy S, $B(t)$ satisfy R_n, $n \geqslant 1$, and $P_j(t)$ be defined by (3.3). Then, if $\alpha > \beta \geqslant 0$, the operator

$$K(t) = \sum_{j=1}^{\infty} P_j(t)\, P_j'(t)$$

is bounded, strongly C^{n-1}, $n \geqslant 1$, on $[0, 2\pi]$, and such that $K(t)^* = -K(t)$.

Remark. The content of this proposition is nontrivial for $\alpha \leqslant 1/2$; see ref. 17.

Actually, the proposition is a consequence of the following technical result.

Lemma 3.2. Assume H_0 satisfies S, $B(t)$ satisfies R_0, and let

$$G(t) = \sum_{j=1}^{\infty} P_j(t) \frac{1}{2\pi i} \oint_{\Gamma_j} A_j(t, \lambda)\, R(t, \lambda)\, d\lambda$$

where

$$\sup_{t \in [0, 2\pi]} \sup_{\lambda \in \Gamma_j} \|A_j(t, \lambda)\| \leqslant \frac{c}{j^{\alpha}}$$

Then, if $\alpha > \beta \geqslant 0$, $G(t)$ is bounded and strongly continuous $\forall t \in [0, 2\pi]$. If, furthermore, $B(t)$ satisfies R_1 and $A_j(t, \lambda)$ is strongly C^1 with

$$\sup_{t \in [0, 2\pi]} \sup_{\lambda \in \Gamma_j} \|A_j'(t, \lambda)\| \leqslant \frac{c}{j^{\alpha}}$$

then $G(t)$ is strongly C^1, $\forall t \in [0, 2\pi]$, and

$$G'(t) = G_1(t) - G_0(t)\, G(t)$$

where $G_0(t)$ and $G_1(t)$ have the same form as $G(t)$ with

$$A_j^0(t, \lambda) = R(t, \lambda)\, B'(t)$$

and

$$A_j^1(t, \lambda) = P_j'(t)\, A_j(t, \lambda) + A_j'(t, \lambda) - A_j(t, \lambda)\, R(t, \lambda)\, B'(t)$$

in place of $A_j(t, \lambda)$.

Remark. The operator $G_0(t)$ coincides with the operator $K(t)$ of the proposition since

$$R'(t, \lambda) = -R(t, \lambda)\, B'(t)\, R(t, \lambda) \tag{3.6}$$

Proof of Proposition 3.1. Since $H_0 + B(t)$ satisfies S uniformly in $t \in [0, 2\pi]$, we have by (3.5)

$$\| R(t, \lambda)|_{\lambda \in \Gamma_j} \| \leqslant \frac{c}{j^\alpha} \tag{3.7}$$

which yields the required bound on $A_j^0(t, \lambda)$. Hence $K(t)$ is bounded provided $\alpha > \beta \geqslant 0$. Now, if $B(t)$ is strongly C^n, the same is true for $R(t, \lambda)$ [see (3.6)] and using the Leibnitz formula or Eq. (2.36) in ref. 14, we get

$$\left\| \left(\frac{\partial}{\partial t}\right)^m R(t, \lambda)|_{\lambda \in \Gamma_j} \right\| \leqslant \frac{c}{j^\alpha}, \qquad \forall m \leqslant n \tag{3.8}$$

Consequently, $A_j^0(t, \lambda)$ is strongly C^{m-1} and

$$\left\| \left(\frac{\partial}{\partial t}\right)^m A_j^0(t, \lambda) \right\| \leqslant \frac{c}{j^\alpha} \tag{3.9}$$

uniformly in $\lambda \in \Gamma_j$ and $t \in [0, 2\pi]$, for all $m \leqslant n-1$. Thus, as is easily checked, the formula of Lemma 3.2 can be iterated since, using (3.4), (3.6), and (3.8),

$$\left\| \left(\frac{\partial}{\partial t}\right)^m P_j(t) \right\| \leqslant \frac{c}{j^\alpha}, \qquad \forall 1 \leqslant m \leqslant n \tag{3.10}$$

The identity $K(t)^* = -K(t)$ results from the self-adjointness of the projectors $P_j(t)$ and the identity

$$0 = \sum_{j=0}^{\infty} P_j'(t) = \sum_{j=0}^{\infty} (P_j'(t)\, P_j(t) + P_j(t)\, P_j'(t)) \quad \blacksquare \tag{3.11}$$

3.2. Adiabatic Formalism

We introduce in this section an iteration scheme which, in the adiabatic context, is also known as superadiabatic renormalization.[13,14] For $t \in [0, 2\pi]$, we start with

$$H_0(t) \equiv H_0 + W(t) \tag{3.12}$$

where H_0 satisfies S and $W(t)$ satisfies R_k, $k \geqslant 1$, so that Lemma 3.1 applies. The spectral projectors of $H_0(t)$ are denoted by

$$P_j^0(t) = -\frac{1}{2\pi i} \oint_{\Gamma_j^0} R_0(t, \lambda)\, d\lambda \tag{3.13}$$

where $R_0(t, \lambda) = (H_0(t) - \lambda)^{-1}$ and Γ_j^0 encircles the spectral set $\sigma_j^0(t)$ in such a way that (3.4) and (3.5) hold (with the obvious change of notation). We define the operator

$$K_0(t) = \sum_{j=1}^{\infty} P_j^0(t)\, P_j^{0\prime}(t) \tag{3.14}$$

which is bounded and strongly C^{k-1} by Proposition 3.1. At the qth step, $k-1 \geqslant q \geqslant 1$, we set

$$H_q(t) \equiv H_0(t) + iK_{q-1}(t) \tag{3.15}$$

which satisfies S as well. Thus we can define its spectral projectors by

$$P_j^q(t) = -\frac{1}{2\pi i} \oint_{\Gamma_j^q} R_q(t, \lambda)\, d\lambda \tag{3.16}$$

where $R_q(t, \lambda) = (H_q(t) - \lambda)^{-1}$ and Γ_j^q encircles the spectral set $\sigma_j^q(t)$ in such a way that (3.4) and (3.5) hold. Similarly, we define

$$K_q(t) = \sum_{j=1}^{\infty} P_j^q(t)\, P_j^{q\prime}(t) \tag{3.17}$$

Using Proposition 3.1 iteratively, we find that $H_q(t)$ is strongly C^{k-q}, whereas $K_q(t)$ is strongly C^{k-q-1}, so that this scheme is well defined provided $q \leqslant k-1$.

Let $V_q(t)$ be the solution of the Schrödinger-like equation for $t \in [0, 2\pi]$

$$iV_q'(t) = (H_q(t) - iK_q(t))\, V_q(t), \qquad V_q(0) = \mathbf{I} \tag{3.18}$$

Lemma 3.3. For any $q \leqslant k-1$, the operator $V_q(t)$ is unitary, maps D into D, and satisfies (3.18) strongly on D. Moreover,

$$V_q(t)\, P_j^q(0) = P_j^q(t)\, V_q(t)$$

for any $t \in [0, 2\pi]$ and any $j = 1, \ldots, \infty$.

Remark. The first part of the lemma is nontrivial for $q = k-1$, since $K_{k-1}(t)$ is not differentiable. The second part generalizes standard results[15,16] which hold for a finite number of projectors.

Corollary 3.1. If $W(t)$ is 2π-periodic and $q \leqslant k-1$,

$$\sigma_{\mathrm{ac}}(V_q(2\pi)) = \varnothing$$

Proof. The operators $H_q(t)$ are 2π-periodic since their construction is local. Hence we have

$$P_j^q(2\pi) = P_j^q(0) \tag{3.19}$$

so that

$$[V_q(2\pi), P_j^q(0)] = 0, \qquad \forall j = 1, \ldots, \infty \tag{3.20}$$

Since the orthogonal subspaces $P_j^q(0)\mathscr{H}$ are finite-dimensional, $V_q(2\pi)$ has pure point spectrum. ∎

Let us now evaluate the difference between $U(2\pi)$ and $V_q(2\pi)$. For $\varphi \in D$ we compute [see (3.12)]

$$\begin{aligned} i(V_q^{-1}(t)\, U(t)\varphi)' &= V_q^{-1}(t)[-H_q(t) + iK_q(t) + H_0(t)]\; U(t)\varphi \\ &= iV_q^{-1}(t)[K_q(t) - K_{q-1}(t)]\; U(t)\varphi \end{aligned} \tag{3.21}$$

Hence

$$U(t) - V_q(t) = V_q(t) \int_0^t ds\; V_q^{-1}(s)[K_q(s) - K_{q-1}(s)]\; U(s) \tag{3.22}$$

In order to apply the Birman–Krein theorem,[2] it remains to show that the trace norm of $U(2\pi) - V_q(2\pi)$ is finite. This will be true if we show that

$$\sup_{t \in [0, 2\pi]} \|K_q(t) - K_{q-1}(t)\|_1 \leqslant c \tag{3.23}$$

where $\|\cdot\|_1$ stands for the trace norm.

3.3. Estimations in the Trace Norm

We first note that

$$K_q(t)-K_{q-1}(t)=\sum_{j=1}^{\infty}\left[P_j^q(t)\,P_j^{q\prime}(t)-P_j^{q-1}(t)\,P_j^{q-1\prime}(t)\right] \tag{3.24}$$

where the operators $P_j^q(t)\,P_j^{q\prime}(t)-P_j^{q-1}(t)\,P_j^{q-1\prime}(t)$ are degenerate. And since $H_q(t)$ and $H_{q-1}(t)$ satisfy the spectral hypothesis S, we have

$$\dim \operatorname{Ran}(P_j^q(t)\,P_j^{q\prime}(t)-P_j^{q-1}(t)\,P_j^{q-1\prime}(t))\leqslant cj^{\beta} \tag{3.25}$$

Our main tools to perform estimations in the trace norm are the following lemmas to be found in ref. 15, p. 521.

Lemma (i). If T is degenerate, $\|T\|_1\leqslant \dim \operatorname{Ran}(T)\,\|T\|$.

Lemma (ii). If $\|T_n-T\|\to 0$ as $n\to\infty$ and $\|T_n\|_1\leqslant M$ uniformly in n, then $\|T\|_1\leqslant M$.

We now state the main proposition of this section.

Proposition 3.2. The projectors $P_j^q(t)$ defined by (3.16) satisfy

$$\left\|\left(\frac{\partial}{\partial t}\right)^n P_j^{q-1}(t)[P_j^q(t)-P_j^{q-1}(t)]\right\|\leqslant\frac{c}{j^{(q+1)\alpha}}$$

for any n and q such that $n+q\leqslant k$, and

$$\left\|\left(\frac{\partial}{\partial t}\right)^n P_j^q(t)[P_j^q(t)\,P_j^{q\prime}(t)-P_j^{q-1}(t)\,P_j^{q-1\prime}(t)]\right\|\leqslant\frac{c}{j^{(q+1)\alpha}}$$

for any n and q such that $n+q+1\leqslant k$.

Corollary 3.2. For $q\leqslant k-1$

$$\|P_j^q(t)\,P_j^{q\prime}(t)-P_j^{q-1}(t)\,P_j^{q-1\prime}(t)\|\leqslant\frac{c}{j^{(q+1)\alpha}}$$

Proof.

$$\begin{aligned}
&P_j^q(t)\,P_j^{q\prime}(t)-P_j^{q-1}(t)\,P_j^{q-1\prime}(t)\\
&\quad=P_j^q(t)[P_j^q(t)\,P_j^{q\prime}(t)-P_j^{q-1}(t)\,P_j^{q-1\prime}(t)]\\
&\qquad+P_j^q(t)\,P_j^{q-1}(t)\,P_j^{q-1\prime}(t)-P_j^{q-1}(t)\,P_j^{q-1}(t)\,P_j^{q-1\prime}(t)\\
&\quad=P_j^q(t)[P_j^q(t)\,P_j^{q\prime}(t)-P_j^{q-1}(t)\,P_j^{q-1\prime}(t)]\\
&\qquad+\{P_j^{q-1}(t)[P_j^q(t)-P_j^{q-1}(t)]\}^*\,P_j^{q-1\prime}(t)
\end{aligned} \tag{3.26}$$

where $\|P_j^{q-1\prime}(t)\|\leqslant c$ [see (3.10)]. ∎

We can now end the proof of Theorem 2.1. On the one hand, using Lemma (i) and the above corollary,

$$\left\| \sum_{j=1}^{n} [P_j^q(t)\, P_j^{q\prime}(t) - P_j^{q-1}(t)\, P_j^{q-1\prime}(t)] \right\|_1 \leqslant c \sum_{j=1}^{n} \frac{j^{\beta}}{j^{(q+1)\alpha}} \leqslant M < \infty \tag{3.27}$$

where M is independent of n, provided $(q+1)\alpha - \beta > 1$. On the other hand,

$$\begin{aligned} &\left\| \sum_{j=1}^{n} [P_j^q(t)\, P_j^{q\prime}(t) - P_j^{q-1}(t)\, P_j^{q-1\prime}(t)] \right. \\ &\qquad \left. - \sum_{j=1}^{\infty} [P_j^q(t)\, P_j^{q\prime}(t) - P_j^{q-1}(t)\, P_j^{q-1\prime}(t)] \right\| \\ &\leqslant c \sum_{j=n+1}^{\infty} \frac{1}{j^{(q+1)\alpha}} \to 0 \qquad \text{as} \quad n \to \infty \end{aligned} \tag{3.28}$$

provided $(q+1)\alpha > 1$. In view of Lemma (ii) and Corollary 3.1, we can conclude by the Birman–Krein theorem[2] that $\sigma_{\mathrm{ac}}(U(2\pi)) = \varnothing$ if $k\alpha - \beta > 1$, since $q \leqslant k-1$. ∎

4. TECHNICALITIES

We present in this final section the proofs of the results stated in the previous section.

Proof of Lemma 3.2. Let us show that $G(t)$ is bounded for any $\alpha > \beta \geqslant 0$. We consider $G(t)$ as an infinite matrix in the orthonormal basis

$$\{\psi_j^{n_j}(t)\}_{j=1,\dots,\infty}^{n_j=1,\dots,\mathrm{mult}(\sigma_j)}, \qquad \text{s.t.} \quad P_j(t)\,\psi_j^{n_j}(t) = \psi_j^{n_j}(t) \tag{4.1}$$

Its matrix elements are given by

$$\begin{aligned} g_{j,k}^{n_j,n_k}(t) &\equiv \langle \psi_j^{n_j}(t) \mid G(t)\, \psi_k^{n_k}(t) \rangle \\ &= \left\langle \psi_j^{n_j}(t) \left| \frac{1}{2\pi i} \oint_{\Gamma_j} A_j(t,\lambda)\, R(t,\lambda)\, d\lambda\, P_k(t)\, \psi_k^{n_k}(t) \right. \right\rangle \end{aligned} \tag{4.2}$$

If $j \neq k$, we have, using the first resolvent identity and the Cauchy formula,

$$\begin{aligned} &\oint_{\Gamma_j} A_j(t,\lambda)\, R(t,\lambda)\, d\lambda\, P_k(t) \\ &\quad = -\frac{1}{2\pi i} \oint_{\Gamma_j} \oint_{\Gamma_k} A_j(t,\lambda)\, R(t,\lambda)\, R(t,\mu)\, d\lambda\, d\mu \\ &\quad = -\frac{1}{2\pi i} \oint_{\Gamma_j} \oint_{\Gamma_k} \left(\frac{A_j(t,\lambda)\, R(t,\lambda)}{\lambda - \mu} - \frac{A_j(t,\lambda)\, R(t,\mu)}{\lambda - \mu} \right) d\lambda\, d\mu \\ &\quad = \frac{1}{2\pi i} \oint_{\Gamma_j} \oint_{\Gamma_k} \frac{A_j(t,\lambda)\, R(t,\mu)}{\lambda - \mu}\, d\lambda\, d\mu \end{aligned} \tag{4.3}$$

As a consequence of the spectral hypothesis S,[10]

$$|\lambda-\mu| \geqslant c((j+1)^\alpha+(j+2)^\alpha+\cdots+k^\alpha) \geqslant c\,|j^{\alpha+1}-k^{\alpha+1}| \tag{4.4}$$

so that by virtue of the hypothesis and our choice of paths (3.4), (3.5) we get the estimate

$$g_{j,k}^{n_j,n_k}(t) \leqslant \frac{c}{|j^{\alpha+1}-k^{\alpha+1}|} \tag{4.5}$$

If $j=k$, we have

$$g_{j,j}^{n_j,n_j}(t) \leqslant \frac{c}{j^\alpha} \tag{4.6}$$

According to the Schur condition (ref. 15, Example 2.3, p. 143),

$$\|G(t)\| \leqslant \max\left(\sup_{j,n_j} \sum_{\substack{k=1\\ n_k}}^{\infty} |g_{j,k}^{n_j,n_k}(t)|, \sup_{k,n_k} \sum_{\substack{j=1\\ n_j}}^{\infty} |g_{j,k}^{n_j,n_k}(t)|\right) \tag{4.7}$$

Thus, since $n_k \leqslant ck^\beta$, we have to show that

$$\sup_j \sum_{\substack{k=1\\ k\neq j}}^{\infty} \frac{k^\beta}{|j^{\alpha+1}-k^{\alpha+1}|} < \infty \qquad \text{and} \qquad \sup_j \frac{j^\beta}{j^\alpha} < \infty \tag{4.8}$$

But it is proven in the Appendix of ref. 10 that, for $\alpha>\beta$,

$$\sum_{\substack{k=1\\ k\neq j}}^{\infty} \frac{k^\beta}{|j^{\alpha+1}-k^{\alpha+1}|} = \mathcal{O}(j^{\beta-\alpha}\ln j) \tag{4.9}$$

so that (4.8) is indeed true provided $\alpha>\beta\geqslant 0$. To prove the strong continuity of $G(t)$, we introduce the strongly continuous approximations

$$G_N(t) = \sum_{j=1}^{N} P_j(t) \frac{1}{2\pi i} \oint_{\Gamma_j} A_j(t,\lambda)\, R(t,\lambda)\, d\lambda \tag{4.10}$$

Applying the Schur condition again, and (4.9), we obtain

$$\lim_{N\to\infty} \sup_{t\in[0,2\pi]} \|G_N(t)-G(t)\| = 0 \tag{4.11}$$

if $\alpha>\beta\geqslant 0$, which shows that $G(t)$ is strongly continuous. Indeed, the conditions to verify are

$$\sup_{j>N}\left(\sum_{\substack{k=1\\k\neq j}}^{\infty}\frac{k^{\beta}}{|j^{\alpha+1}-k^{\alpha+1}|}+\frac{j^{\beta}}{j^{\alpha}}\right)\xrightarrow{N\to\infty} 0 \tag{4.12}$$

$$\sup_{k}\sum_{\substack{j=N+1\\j\neq k}}^{\infty}\frac{j^{\beta}}{|j^{\alpha+1}-k^{\alpha+1}|}+\sup_{j>N}\frac{j^{\beta}}{j^{\alpha}}\xrightarrow{N\to\infty} 0 \tag{4.13}$$

According to (4.9), (4.12) is true if $\alpha>\beta\geqslant 0$ and the sum in (4.13) can be estimated by

$$\begin{aligned}&\sup_{k\leqslant N}\sum_{\substack{j=N+1\\j\neq k}}^{\infty}\frac{j^{\beta}}{|j^{\alpha+1}-k^{\alpha+1}|}+\sup_{k>N}\sum_{\substack{j=N+1\\j\neq k}}^{\infty}\frac{j^{\beta}}{|j^{\alpha+1}-k^{\alpha+1}|}\\&\leqslant\sum_{j=N+1}^{\infty}\frac{j^{\beta}}{|j^{\alpha+1}-N^{\alpha+1}|}+\sup_{k>N}\sum_{\substack{j=N+1\\j\neq k}}^{\infty}\frac{j^{\beta}}{|j^{\alpha+1}-k^{\alpha+1}|}\end{aligned} \tag{4.14}$$

where both terms are $\mathcal{O}(N^{\beta-\alpha}\ln N)$ again.

To consider the differentiability of $G(t)$ when $B(t)$ and $A_j(t,\lambda)$ are strongly C^1, we also introduce the projector

$$\Pi_N(t)=\mathbf{I}-\sum_{j=1}^{N}P_j(t) \tag{4.15}$$

By (3.6) and a standard application of the Cauchy formula, we can write

$$\Pi'_N(t)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}i\,d\eta\,R(t,\xi_N+i\eta)\,B'(t)\,R(t,\xi_N+i\eta) \tag{4.16}$$

where ξ_N lies on the real axis between σ_N and σ_{N+1}. Hence

$$\|\Pi'_N(t)\|\leqslant c\,\|B'(t)\|\int_{-\infty}^{\infty}d\eta\,\frac{1}{cN^{2\alpha}+\eta^2}\leqslant\frac{c}{N^{\alpha}}\int_{-\infty}^{\infty}dy\,\frac{1}{1+y^2}\leqslant\frac{c}{N^{\alpha}} \tag{4.17}$$

Let us compute

$$\begin{aligned}&\frac{\partial}{\partial t}\left\{\sum_{j=1}^{N}P_j(t)\frac{1}{2\pi i}\oint_{\Gamma_j}A_j(t,\lambda)\,R(t,\lambda)\,d\lambda\right\}\\&=\sum_{j=1}^{N}[P'_j(t)\,P_j(t)+P_j(t)\,P'_j(t)]\frac{1}{2\pi i}\oint_{\Gamma_j}A_j(t,\lambda)\,R(t,\lambda)\,d\lambda\\&\quad+\sum_{j=1}^{N}P_j(t)\frac{1}{2\pi i}\oint_{\Gamma_j}[A'_j(t,\lambda)-A_j(t,\lambda)\,R(t,\lambda)\,B'(t)]\,R(t,\lambda)\,d\lambda\end{aligned}$$

$$= \sum_{j=1}^{N} P_j(t) \frac{1}{2\pi i} \oint_{\Gamma_j} [P_j'(t)\, A_j(t, \lambda) + A_j'(t, \lambda) - A_j(t, \lambda)\, R(t, \lambda)\, B'(t)]$$
$$\times R(t, \lambda)\, d\lambda$$
$$+ \sum_{k=1}^{N} P_k'(t)\, P_k(t) \sum_{j=1}^{N} P_j(t) \oint_{\Gamma_j} A_j(t, \lambda)\, R(t, \lambda)\, d\lambda \tag{4.18}$$

Now, using the definition (4.15),

$$\sum_{j=1}^{N} P_j'(t) + \Pi_N'(t) = \sum_{j=1}^{N} P_j'(t)\, P_j(t) + P_j(t)\, P_j'(t) + \Pi_N'(t) \equiv 0 \tag{4.19}$$

so that

$$\sum_{j=1}^{N} P_j'(t)\, P_j(t) = -\sum_{j=1}^{N} P_j(t)\, P_j'(t) - \Pi_N'(t) \tag{4.20}$$

Consequently, the last term of (4.18) can be written as

$$- \sum_{k=1}^{N} P_k(t)\, P_k'(t)\, G_N(t) - \Pi_N'(t)\, G_N(t)$$
$$\equiv -G_{0,N}(t)\, G_N(t) - \Pi_N'(t)\, G_N(t) \tag{4.21}$$

where $G_{0,N}$ approximates $G_0(t)$ as in (4.10) (see the remark below Lemma 3.2). We have thus proved the formula, with a similar notation,

$$G_N'(t) = G_{1,N}(t) - G_{0,N}(t)\, G_N(t) - \Pi_N'(t)\, G_N(t) \tag{4.22}$$

where $G_{1,N}(t)$ tends to $G_1(t)$ in norm and uniformly in $t \in [0, 2\pi]$ by our hypothesis on $\|A_j'(t, \lambda)\|$. On the other hand,

$$G(s)\,|_{t_0}^{t} = G_N(s)\,|_{t_0}^{t} + [G(s) - G_N(s)]\,|_{t_0}^{t}$$
$$= \int_{t_0}^{t} G_N'(s)\, ds + [G(s) - G_N(s)]\,|_{t_0}^{t}$$
$$= \int_{t_0}^{t} [G_{1,N}(s) - G_{0,N}(s)\, G_N(s) - \Pi_N'(s)\, G_N(s)]\, ds$$
$$+ [G(s) - G_N(s)]\,|_{t_0}^{t} \tag{4.23}$$

where the integrand tends to $G_1(s) - G_0(s)\, G(s)$ in norm and uniformly in $s \in [0, 2\pi]$, by (4.11) and (4.17). Thus we can take the limit $N \to \infty$ inside the integral in the above equation, which eventually yields

$$G(t) - G(t_0) = \int_{t_0}^{t} G_1(s) - G_0(s)\, G(s)\, ds \quad \blacksquare \tag{4.24}$$

Proof of Lemma 3.3. In order to simplify the notation, we drop the indices q so that we are led to consider

$$H(t) = H_0 + B(t) \tag{4.25}$$

where H_0 satisfies S with $\alpha > \beta \geqslant 0$, $B(t)$ satisfies R_1, and the operator

$$K(t) = \sum_{j=1}^{\infty} P_j(t)\, P_j'(t) \tag{4.26}$$

where $P_j(t)$ is defined by (3.3). We first have to show that there exists a unitary operator $V(t)$ strongly C^1 on D, mapping D into D such that for any $\varphi \in D$ and $t \in \mathbf{R}$

$$iV'(t)\varphi = [H(t) - iK(t)]\, V(t)\varphi, \qquad V(0) = \mathbf{I} \tag{4.27}$$

Note that as $K(t)$ is strongly continuous only, we cannot invoke Theorem X70 of ref. 19 directly. We shall instead make use of a theorem of perturbation of evolution operators, Theorem 3.4 p. 198, in ref. 16. Since $H'(t) = B'(t)$ is bounded, there exists a unitary $U(t)$ with the above properties such that

$$iU'(t)\varphi = H(t)\, U(t)\varphi, \qquad U(0) = \mathbf{I} \tag{4.28}$$

if $\varphi \in D$. Then, according to the above-mentioned theorem, there exists a unitary $V(t)$ associated with (4.27) possessing the required properties, provided the perturbation $K(t)$ and the operator $H(t)\, K(t)\, R(t, 0)$ are bounded and strongly continuous. Let $\varphi \in D$. We have for any $j \geqslant 1$

$$H(t)\, P_j(t)\varphi = P_j(t)\, H(t)\varphi \tag{4.29}$$

so that (applying Lemma 1.3, p. 178, ref. 16, for example)

$$[H'(t)\, P_j(t) + H(t)\, P_j'(t)]\varphi = [P_j'(t)\, H(t) + P_j(t)\, H'(t)]\varphi \tag{4.30}$$

Hence we compute, using (4.25) and the completeness of the projectors $P_j(t)$,

$$\begin{aligned}
&H(t)\, K(t)\, R(t, 0)\\
&\quad = \sum_{j=1}^{\infty} P_j(t)\, H(t)\, P_j'(t)\, R(t, 0)\\
&\quad = \sum_{j=1}^{\infty} P_j(t)[P_j'(t)\, H(t) + P_j(t)\, B'(t) - B'(t)\, P_j(t)]\, R(t, 0)\\
&\quad = \sum_{j=1}^{\infty} [P_j(t)\, P_j'(t) + P_j(t)\, B'(t)\, R(t, 0) - P_j(t)\, B'(t)\, R(t, 0)\, P_j(t)]\\
&\quad = K(t) + B'(t)\, R(t, 0) - \sum_{j=1}^{\infty} P_j(t)\, B'(t)\, R(t, 0)\, P_j(t)
\end{aligned} \tag{4.31}$$

The last operator is bounded,

$$\begin{aligned}
&\left\| \sum_{j=1}^{\infty} P_j(t)\, B'(t)\, R(t,0)\, P_j(t)\varphi \right\|^2 \\
&\quad = \sum_{j=1}^{\infty} \|P_j(t)\, B'(t)\, R(t,0)\, P_j(t)\varphi\|^2 \\
&\quad \leqslant \sum_{j=1}^{\infty} \|P_j(t)\, B'(t)\, R(t,0)\|^2\, \|P_j(t)\varphi\|^2 \\
&\quad \leqslant \|B'(t)\, R(t,0)\|^2 \sum_{j=1}^{\infty} \|P_j(t)\varphi\|^2 \\
&\quad = \|B'(t)\, R(t,0)\|^2\, \|\varphi\|^2
\end{aligned} \tag{4.32}$$

and strongly continuous since it is a sum of strongly continuous operators converging uniformly in $t \in [0, 2\pi]$. Indeed, it is sufficient to note that

$$\begin{aligned}
&\sum_{j=N+1}^{\infty} \|P_j(t)\varphi\|^2 \\
&\quad = \|(\mathbf{I} - \Pi_N(t))\varphi\|^2 \\
&\quad = \left\| \left[\mathbf{I} - \Pi_N(0) - \int_0^t ds\, \Pi'_N(s) \right] \varphi \right\|^2 \\
&\quad \leqslant \|[\mathbf{I} - \Pi_N(0)]\varphi\|^2 + \frac{C}{N^\alpha}\, \|\varphi\|^2 \xrightarrow{N \to \infty} 0
\end{aligned} \tag{4.33}$$

uniformly in $t \in [0, 2\pi]$, according to (4.17).

To prove the intertwining property

$$V(t)\, P_j(0) = P_j(t)\, V(t), \qquad \forall t \in [0, 2\pi] \quad \text{and} \quad \forall j \geqslant 1 \tag{4.34}$$

we approximate $V(t)$ by evolution operators $V_N(t)$, $N = 1,..., \infty$, associated with the equations

$$iV'_N(t)\varphi = [H(t) - iK_N(t) - i\Pi_N(t)\, \Pi'_N(t)]\, V_N(t)\varphi, \qquad V_N(0) = \mathbf{I} \tag{4.35}$$

where $\varphi \in D$ and $t \in \mathbf{R}$ and

$$K_N(t) = \sum_{j=1}^{N} P_j(t)\, P'_j(t) \equiv G_{0,N}(t) \tag{4.36}$$

As above, we deduce that $V_N(t)$ are unitary, strongly C^1 on D, and map D into D since $\Pi_N(t)$ commutes with $H(t)$ and

$$\sum_{j=1}^{N} P_j(t) + \Pi_N(t) = \mathbf{I} \tag{4.37}$$

Moreover, since we deal with a finite number of projectors, we have the standard property (see, e.g., ref. 16, §3, Chapter IV)

$$\begin{aligned} V_N(t)\, P_j(0) &= P_j(t)\, V_N(t) \qquad \forall j = 1,..., N \\ V_N(t)\, \Pi_N(0) &= \Pi_N(t)\, V_N(t) \qquad \forall t \in [0, 2\pi] \end{aligned} \tag{4.38}$$

We compute

$$\begin{aligned} V(t)\, P_j(0) - P_j(t)\, V(t) = {} & [V(t) - V_N(t)]\, P_j(0) - P_j(t)[V(t) - V_N(t)] \\ & + [V_N(t)\, P_j(0) - P_j(t)\, V_N(t)] \end{aligned} \tag{4.39}$$

where the last bracket vanishes if $j \leqslant N$. Then, for any $\varphi \in D$,

$$i\frac{\partial}{\partial t}[V_N^{-1}(t)\, V(t)\varphi] = iV_N^{-1}(t)[K_N(t) - K(t) + \Pi_N(t)\, \Pi'_N(t)]\, V(t)\varphi \tag{4.40}$$

so that

$$\|V(t) - V_N(t)\| \leqslant \int_0^t ds\, \|K_N(s) - K(s) + \Pi_N(s)\, \Pi'_N(s)\| \tag{4.41}$$

But the integrand tends to zero uniformly in s as N tends to infinity [see (4.11) and (4.17)] so that

$$\lim_{N\to\infty} \|V(t) - V_N(t)\| = 0 \qquad \forall t \in [0, 2\pi] \tag{4.42}$$

Consequently, we deduce from (4.39) that $V(t)\, P_j(0) - P_j(t)\, V(t) \equiv 0$ for any $j \geqslant 1$ and any $t \in [0, 2\pi]$. ∎

Proof of Proposition 3.2. The proof is by induction. Let us recall that $W(t)$ is strongly C^k and $q \leqslant k-1$. We first note that, at the cost of a possible relabeling, we can assume that for j large enough the spectral sets $\sigma_j^s(t)$, $s = 0, 1,..., q+1$, can be surrounded by the same path Γ_j such that

$$\begin{aligned} |\Gamma_j| &\leqslant cj^\alpha \\ \operatorname{dist}\left(\Gamma_j, \bigcup_{j=1}^{\infty} \sigma_j^s(t)\right) &\geqslant cj^\alpha, \qquad s = 0, 1,..., q+1 \end{aligned} \tag{4.43}$$

Indeed, $H_s(t) = H_0(t) + iK_{s-1}(t)$, where

$$\max_{s=1,\dots,q+1} \sup_{t\in[0,2\pi]} \|K_{s-1}(t)\| \leqslant c \tag{4.44}$$

We set, using the second resolvent identity,

$$\begin{aligned} T_j^q(t) &\equiv P_j^{q+1}(t) - P_j^q(t) \\ &= -\frac{1}{2\pi i}\oint_{\Gamma_j} [R_{q+1}(t,\lambda) - R_q(t,\lambda)]\, d\lambda \\ &= \frac{1}{2\pi}\oint_{\Gamma_j} R_q(t,\lambda)[K_q(t) - K_{q-1}(t)]\, R_{q+1}(t,\lambda)\, d\lambda \end{aligned} \tag{4.45}$$

Lemma 4.1.

(a) $T_j^q(t) = P_j^{q+1}(t)\, T_j^q(t) + T_j^q(t)\, P_j^q(t)$

(b) $P_j^{q+1}(t)\, T_j^q(t) = P_j^{q+1}(t)\, P_j^q(t)\, T_j^q(t)[\mathbf{I} - T_j^q(t)]^{-1}$

Proof. (a) Since the operators $P_j^s(t)$ are projectors, we can write

$$\begin{aligned} P_j^{q+1}(t) &= [P_j^q(t) + T_j^q(t)]^2 \\ &= P_j^q(t) + P_j^q(t)\, T_j^q(t) + T_j^q(t)\, P_j^q(t) + T_j^q(t)\, T_j^q(t) \end{aligned} \tag{4.46}$$

Hence

$$T_j^q(t) = [P_j^q(t) + T_j^q(t)]\, T_j^q(t) + T_j^q(t)\, P_j^q(t) \tag{4.47}$$

(b) For the same reason we have

$$\begin{aligned} P_j^{q+1}(t)\, T_j^q(t) &= P_j^{q+1}(t)[P_j^q(t) + T_j^q(t)]\, T_j^q(t) \\ &= P_j^{q+1}(t)\, P_j^q(t)\, T_j^q(t) + P_j^{q+1}(t)\, T_j^q(t)\, T_j^q(t) \end{aligned} \tag{4.48}$$

so that

$$P_j^{q+1}(t)\, T_j^q(t)[\mathbf{I} - T_j^q(t)] = P_j^{q+1}(t)\, P_j^q(t)\, T_j^q(t) \tag{4.49}$$

From (4.45) and (4.43) we obtain the estimate

$$\begin{aligned} \|T_j^q(t)\| &\leqslant c\left\|\oint_{\Gamma_j} R_q(t,\lambda)[K_q(t) - K_{q-1}(t)]\, R_{q+1}(t,\lambda)\, d\lambda\right\| \\ &\leqslant c\, |\Gamma_j|\, \frac{1}{j^{2\alpha}}\, \|K_q(t) - K_{q-1}(t)\| \leqslant \frac{c}{j^\alpha} \end{aligned} \tag{4.50}$$

which shows that for j large enough the operator $[\mathbf{I} - T_j^q(t)]$ is invertible. ∎

Remark. We can deduce easily from this lemma and the self-adjointness of $P_j^\alpha(t)$ the following estimate for j large enough:

$$\|T_j^q(t)\| \leqslant c\,\|P_j^q(t)\,T_j^q(t)\| \tag{4.51}$$

which yields the sharper estimate on $\|T_j^q(t)\|$

$$\|T_j^q(t)\| \leqslant \frac{c}{j^{(q+2)\alpha}} \tag{4.52}$$

provided Proposition 3.2 is true.

Let us assume the proposition holds for the index q and let us check its validity for $q+1$. Thus we have to estimate for any n such that $n+q+1 \leqslant k$ [see (4.45)]

$$\begin{aligned}\left(\frac{\partial}{\partial t}\right)^n P_j^q(t)\,T_j^q(t) = \frac{1}{2\pi}\oint_{\Gamma_j}\left(\frac{\partial}{\partial t}\right)^n R_q(t,\lambda)\,P_j^q(t)[K_q(t)-K_{q-1}(t)]&\\ \times R_{q+1}(t,\lambda)\,d\lambda&\end{aligned} \tag{4.53}$$

We compute

$$\begin{aligned}&P_j^q(t)[K_q(t)-K_{q-1}(t)]\\ &\quad = P_j^q(t)[P_j^q(t)\,K_q(t) - P_j^{q-1}(t)\,K_{q-1}(t)]\\ &\qquad + P_j^q(t)\,P_j^{q-1}(t)\,K_{q-1}(t) - P_j^q(t)\,K_{q-1}(t)\\ &\quad = P_j^q(t)[P_j^q(t)\,P_j^{q\prime}(t) - P_j^{q-1}(t)\,P_j^{q-1\prime}(t)]\\ &\qquad - P_j^q(t)\,T_j^{q-1}(t)\,K_{q-1}(t)\end{aligned} \tag{4.54}$$

Using property (b), we get

$$\begin{aligned}&\left(\frac{\partial}{\partial t}\right)^n P_j^q(t)\,T_j^q(t)\\ &\quad = \frac{1}{2\pi}\oint_{\Gamma_j}\left(\frac{\partial}{\partial t}\right)^n R_q(t,\lambda)\,P_j^q(t)[P_j^q(t)\,P_j^{q\prime}(t) - P_j^{q-1}(t)\,P_j^{q-1\prime}(t)]\\ &\qquad \times R_{q+1}(t,\lambda)\,d\lambda\\ &\qquad - \frac{1}{2\pi}\oint_{\Gamma_j}\left(\frac{\partial}{\partial t}\right)^n R_q(t,\lambda)\,P_j^q(t)\,P_j^{q-1}(t)\\ &\qquad \times T_j^{q-1}(t)[\mathbf{I}-T_j^{q-1}(t)]^{-1}\,K_{q-1}(t)\,R_{q+1}(t,\lambda)\,d\lambda\end{aligned} \tag{4.55}$$

By the Leibnitz formula or Eq. (2.36) in ref. 14, the nth time derivative of $R_s(t, \lambda)$ for $s = 0,..., q+1$ is given by a sum of products of operators $R_s(t, \lambda)$ and

$$\left(\frac{\partial}{\partial t}\right)^l H_s(t) = \left(\frac{\partial}{\partial t}\right)^l W(t) + i\left(\frac{\partial}{\partial t}\right)^l K_{s-1}(t), \qquad l = 1,..., n \tag{4.56}$$

Hence, provided $n + q + 1 \leqslant k$ (see Section 3.2), and making use of (4.43), we get

$$\left\|\left(\frac{\partial}{\partial t}\right)^n R_s(t, \lambda)|_{\lambda \in \Gamma_j}\right\| \leqslant \frac{c}{j^\alpha} \qquad \text{and} \qquad \left\|\left(\frac{\partial}{\partial t}\right)^n P_j^s(t)\right\| \leqslant \frac{c}{j^\alpha}, \qquad n \geqslant 1 \tag{4.57}$$

for $s = 0,..., q+1$. The same argument can be applied to estimate the nth time derivative of $[\mathbf{I} - T_j^{q-1}(t)]^{-1}$. Indeed, for j large enough,

$$[\mathbf{I} - T_j^{q-1}(t)]^{-1\prime} = [\mathbf{I} - T_j^{q-1}(t)]^{-1} T_j^{q-1\prime}(t)[\mathbf{I} - T_j^{q-1}(t)]^{-1} \tag{4.58}$$

and

$$\left(\frac{\partial}{\partial t}\right)^l T_j^{q-1}(t) = \frac{1}{2\pi}\oint_{\Gamma_j}\left(\frac{\partial}{\partial t}\right)^l R_{q-1}(t, \lambda)[K_{q-1}(t) - K_{q-2}(t)]\, R_q(t, \lambda)\, d\lambda \tag{4.59}$$

so that

$$\left\|\left(\frac{\partial}{\partial t}\right)^n [\mathbf{I} - T_j^{q-1}(t)]^{-1}\right\| \leqslant c, \qquad \forall n + q + 1 \leqslant k \tag{4.60}$$

Thus, invoking the induction hypothesis and (4.57), we get

$$\left\|\left(\frac{\partial}{\partial t}\right)^n P_j^q(t)\, T_j^q(t)\right\| \leqslant \frac{c}{j^{(q+2)\alpha}}, \qquad \forall n + q + 1 \leqslant k \tag{4.61}$$

Now, if $q + 2 \leqslant k$, we compute

$$\begin{aligned} &P_j^{q+1}(t)[P_j^{q+1}(t)\, P_j^{q+1\prime}(t) - P_j^q(t)\, P_j^{q\prime}(t)] \\ &\quad = P_j^{q+1}(t)[T_j^q(t)\, P_j^{q\prime}(t) + P_j^{q+1}(t)\, T_j^{q\prime}(t)] \\ &\quad = [P_j^{q+1}(t)\, T_j^q(t)]\, P_j^{q\prime}(t) + P_j^{q+1}(t)[P_j^{q+1}(t)\, T_j^q(t)]' \\ &\qquad - P_j^{q+1}(t)\, P_j^{q+1\prime}(t)\, T_j^q(t) \end{aligned} \tag{4.62}$$

where the last term can be written as

$$P_j^{q+1}(t)\, P_j^{q+1\prime}(t)\, T_j^q(t) = P_j^{q+1\prime}(t)[T_j^q(t)\, P_j^q(t)] \tag{4.63}$$

using the identity $Q(t)\,Q'(t)\,Q(t) \equiv 0$ for any projector $Q(t)$ and Lemma 4.1(a). Then, by Lemma 4.1(b),

$$\begin{aligned}
&P_j^{q+1}(t)[P_j^{q+1}(t)\,P_j^{q+1\prime}(t) - P_j^q(t)\,P_j^{q\prime}(t)] \\
&\quad = P_j^{q+1}(t)[P_j^q(t)\,T_j^q(t)][\mathbf{I} - T_j^q(t)]^{-1}\,P_j^{q\prime}(t) \\
&\quad\quad + P_j^{q+1}(t)\{P_j^{q+1}(t)[P_j^q(t)\,T_j^q(t)][\mathbf{I} - T_j^q(t)]^{-1}\}' \\
&\quad\quad - P_j^{q+1\prime}(t)[P_j^q(t)\,T_j^q(t)]^*
\end{aligned} \tag{4.64}$$

Finally, considering the Leibnitz formula again, (4.61), (4.60), (4.57), and

$$\left(\frac{\partial}{\partial t}\right)^l [P_j^q(t)\,T_j^q(t)]^* = \left\{\left(\frac{\partial}{\partial t}\right)^l [P_j^q(t)\,T_j^q(t)]\right\}^* \tag{4.65}$$

we obtain similarly

$$\left\|\left(\frac{\partial}{\partial t}\right)^n P_j^{q+1}(t)[P_j^{q+1}(t)\,P_j^{q+1\prime}(t) - P_j^q(t)\,P_j^{q\prime}(t)]\right\| \leqslant \frac{c}{j^{(q+2)\alpha}} \tag{4.66}$$

for any n such that $n+q+2 \leqslant k$. Finally, the induction hypothesis is readily verified for $q = 1$ with $n+1 \leqslant k$ on

$$\begin{aligned}
P_j^0(t)\,T_j^0(t) &= \frac{1}{2\pi}\oint_{\Gamma_j} R_0(t,\lambda)\,P_j^0(t)\,K_0(t)\,R_1(t,\lambda)\,d\lambda \\
&= \frac{1}{2\pi}\oint_{\Gamma_j} R_0(t,\lambda)\,P_j^0(t)\,P_j^{0\prime}(t)\,R_1(t,\lambda)\,d\lambda
\end{aligned} \tag{4.67}$$

[see (4.57) and (4.43)], which implies, as above, that it is satisfied by

$$P_j^1(t)[P_j^1(t)\,P_j^{1\prime}(t) - P_j^0(t)\,P_j^{0\prime}(t)] \tag{4.68}$$

with $n+2 \leqslant k$. ▮

5. CONCLUSION

The spectrum of the Floquet operator associated with time-periodic perturbations $W(t) = W(t+2\pi)$ of discrete Hamiltonians H_0 has been considered. More precisely, let $U(t)$ denote the unitary evolution operator solution of the time-dependent Schrödinger equation with Hamiltonian $H_0 + W(t)$. Then the spectra of the Floquet operator and of $U(2\pi)$ are of equivalent nature. It was shown in particular that if the gap between successive eigenvalues λ_j of the unperturbed Hamiltonian H_0 grows as $\lambda_j - \lambda_{j-1} \simeq j^\alpha$ and the multiplicity of λ_j grows as j^β with $\alpha > \beta \geqslant 0$ as j tends to infinity, then the absolutely continuous spectrum of $U(2\pi)$ is empty provided the perturbation $W(t)$ is $[(1+\beta)/\alpha]+1$ times differentiable, where $[\cdot]$ denotes the integer part. We have used recently developed tools in adiabatic theory to construct unitary approximations V_q, $q=1, 2,\ldots$, of $U(2\pi)$ by iteration [provided $W(t)$ is smooth enough] such that the spectrum of V_q is pure point for any q and $U(2\pi)-V_q$ is trace class for q large enough. The conclusion was reached by classical results on the stability of absolutely continuous spectrum of unitary operators under trace-class perturbations.

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